Challenge of Champions Test 2007 Solutions

1. How many minutes are in 75% of a day?

There are $24 \times 60 \times 3/4 = 1080$ minutes.

2. Sammy will roll two standard six-sided dice once. What is the probability that the two numbers rolled will be the same?

There are 36 possible outcomes of rolling and there are 6 outcomes satisfying the conditions of the problem. Hence probability is $6/36 = 1/6$.

3. Find the largest integer $n$ such that $n^{200} < 6^{300}$.

Taking the 100th root of both sides, the inequality becomes $n^2 < 6^3 = 216$, giving the largest $n$ equal to 14.

4. Find the number of digits of $4^{15} \cdot 5^{25}$.

$4^{15} \cdot 5^{25} = 2^{30} \cdot 5^{25} = 2^5 \cdot 2^{25} \cdot 5^{25} = 2^5 \cdot 10^{25}$ that is the number has 27 digits.

5. How many positive integers less than 1000 can be written using only digits 0, 2 and 7?

There are 2 positive, one-digit integers, $2 \times 3 = 6$ two-digit integers, and $2 \times 3 \times 3 = 18$ three-digit integers. Hence the total number of integers in question is 26.
6. A total of 3000 marbles of three different colors red, yellow, and green are placed in a line. The first three marbles are red, followed by four yellow, followed by six green, followed by three red, followed by four yellow, followed by six green, ... What color is 2007th marble?

The pattern of 13 consecutive marbles repeats, so that the 14th marble is the same as the 1st one, 15th marble is the same as the 2nd one etc. Since 2007 = 13 \cdot 154 + 5, it follows that the 2007th marble is the same as the 5th one, so it is yellow.

7. Circles with centers (3, 1) and (15, 6) are both tangent to the x-axis. What is the distance between the closest points on the two circles?

Consider a right triangle with centers A(3, 1), B(15, 1) and C(15, 6). Its legs have lengths 5 and 12, so the hypotenuse is 13. The closest distance between the circles is then 13 − 1 − 6 = 6.

8. A block of cheese 6 inches by 4 inches by 3 inches is coated with a very thin sheet of wax paper. If the cheese block is cut into 576 half inch cubes, how many cubes have no wax on them?

Looking at the 3 \times 4 sides, we see that there are 24 cubes that would not get wax paper from the other four sides. There are 12 \times 24 = 288 such cubes. 48 of them however will get wax paper, and thus the number of cubes that are wax-free is 288 − 48 = 240.

9. Are there two numbers whose sum and product are both 12? If so, what are the numbers?

Let the numbers be a and b. Then ab = 12, and a + b = 12, giving a + 12/a = 12, or a^2 − 12a + 12 = 0. By the quadratic formula we get a = 6 + 2\sqrt{6}, b = 6 − 2\sqrt{6}.
10. Find the area of the quadrilateral with endpoints (1, 1), (7, 2), (5, 4), and (2, 5).

Consider the points $A(1, 7)$, $B(7, 2)$, $C(5, 4)$, $D(7, 5)$ and $P(7, 1)$, $R(7, 5)$ and $S(1, 5)$, and the rectangle $APRS$ whose area is $6 \times 4 = 24$. To find the area of our quadrilateral $ABCD$ we subtract the areas of triangles $APB$, $BCR$, $RDC$ and $ADS$ from the area of the rectangle $APRS$. We have the areas of triangles $|APB| = (1/2)6 \times 1 = 3$, $|BCR| = (1/2)3 \times 2 = 3$, $|RDC| = (1/2)5 \times 1 = 5/2$, $|ADS| = (1/2)4 \times 1 = 2$. Hence the area of the quadrilateral $|ABCD| = 24 - (3 + 3 + 2 + 5/2) = 13.5$