1. The sides of a triangle are in the ratio of 2 : 3 : 4. Find the cosine of the smallest angle.

Solution: Let the side lengths of the triangle be 2a, 3a and 4a for some positive constant a. The smallest angle is opposite of the smallest side, that is opposite to 2a. Denote this angle by A. Then by the Law of Cosines one has

\[(2a)^2 = (3a)^2 + (4a)^2 - 2(3a)(4a) \cos A\]

giving \(\cos A = \frac{21}{24} = \frac{7}{8}\).

2. Find all solutions to

\[x^{\log x} = \frac{x^3}{100}\]

where \(\log x = \log_{10} x\).

Solution: Let \(x > 0\), and take logarithms of both sides

\[\log(x^{\log x}) = \log(x^3/100),\]

so that

\[(\log x)^2 - 3\log x + 2 = 0\]

and factoring gives \(\log x = 1\) or \(\log x = 2\), that is \(x = 10\) or \(x = 100\).

3. Let \(p, q\) and \(r\) be integers such that

\[f(x) = x^4 + 4x^3 + 6px^2 + 4qx + r\]

is divisible by \(x^3 + 3x^2 + 9x + 3\). Find \(p + q + r\).

Solution: Let \(g(x) = x^3 + 3x^2 + 9x + 3\). The conditions of the problem require that \(f(x) = g(x)(x + a)\), for some real number \(a\). Multiplying out and comparing the coefficients of some power of \(x\) we obtain: \(a = 1, 12 = 6p, 12 = 4q, 3 = r\), so \(p + q + r = 2 + 3 + 3 = 8\).
4. If $\sin x + \cos x = -1/5$ and $3\pi/4 \leq x \leq \pi$, find $\cos(2x)$.

Solution: Squaring $\sin x + \cos x = -1/5$ one gets $\sin(2x) = -24/25$, hence $\cos(2x) = \pm \sqrt{1 - (-24/25)^2} = \pm 7/25$. Since $x \in [3\pi/4, \pi]$, then $2x \in [3\pi/2, 2\pi]$ so $\cos x > 0$ and therefore $\cos(2x) = 7/25$.

5. Show that $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots - \frac{1}{2005} + \frac{1}{2006} > \frac{1}{5}$.

Solution: The left hand side can be written in the form $(1/2 - 1/3) + (1/4 - 1/5) + r$, where $r$ is a positive number. Hence it is bigger then $(1/2 - 1/3) + (1/4 - 1/5) = 1/6 + 1/10 = 13/60 > 12/60 = 1/5$.

6. Let $f(x) = ax^2 + bx + c$ where $a$, $b$ and $c$ are integers and $5$ divides $f(n)$ for any integer $n$. Show that $a$, $b$ and $c$ are each divisible by $5$.

Solution: Since $5|f(0) = c$, $5|f(1) = a + b + c$, $5|f(-1) = a - b + c$, we conclude then $5|a + b$ and $5|a - b$ so $5|(a + b) - (a - b)$ so $5|2a$ or $5|a$ and also $5|b$.

7. The real numbers $a$, $b$ and $c$ are such that $a + b + c = 5$ and

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{2}{3}.$$ 

Find the value of

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a}.$$ 

Solution: Let $S = c/(a+b) + a/(b+c) + b/(c+a)$. Then $S + 3 = (c + a + b)/(a+b) + (a + b + c)/(b+c) + (b + c + a)/(c+a) = (a + b + c)(1/(a+b) + 1/(b+c) + 1/(c+a))$ so that $S + 3 = 5 \cdot (2/3) = 10/3$. Hence $S = 10/3 - 3 = 1/3$.

8. If a tetrahedron (a solid with 4 equilateral triangles as faces) has edge length 1 for each edge. What is the height of the tetrahedron?

Solution: The height $a$ of the tetrahedron joins the vertex with the orthocenter (where the medians intersect) of the opposite side. The orthocenter of the side divides the height $h$ of the side in the ratio $2:1$. By Pythagorean Theorem $1^2 = a^2 + ((2/3)h)^2$, where $h = \sqrt{3}/2$. Hence $a = \sqrt{2}/3$. 

9. A grasshopper jumps along a number line starting at the point 0. The first jump takes him 1 cm, the second 2 cm, the third 3 cm and so on. Each jump takes him to the right or to the left. Can the grasshopper return to the point 0 on the 73rd jump?

Solution: The grasshopper lands on numbers with the following parity pattern after one jump odd, after second jump odd, after third jump even, after fourth jump even, after fifth jump odd, after sixth jump odd, after seventh jump even, and so on. Hence the grasshopper must be on an odd number after the 73 jump so this number can’t be 0.

10. Can one find 2006 positive integers whose sum is equal to its product?

Solution: The answer is yes. Let us take for instance \( x_1 = x_2 = \cdots = x_{2004} = 1, x_{2005} = a \) and \( x_{2006} = b \), where \( a \) and \( b \) are positive integers. Then we need to have 2004 + \( a + b = ab \) or 2005 = \( ab - a - b + 1 = (a-1)(b-1) \). Such integers exist: for example let \( a - 1 = 5, b - 1 = 401 \), giving \( a = 6 \) and \( b = 402 \).