Challenge of Champions Test 2005: Solutions

1. What is the value of $k$ if

$$2^{2007} - 2^{2006} - 2^{2005} + 2^{2004} = k \cdot 2^{2004} ?$$

Since the exponent of the side with the unknown term $k$ is 2004, we will express all the terms using this exponent. This gives

$$k \cdot 2^{2004} = 2^{2004} \cdot 2^3 - 2^{2004} \cdot 2^2 - 2^{2004} \cdot 2 + 2^{2004} = (8-4-2+1)2^{2004} = 3 \cdot 2^{2004},$$

so $k = 3$.

2. Joe rolls an eight-sided die and Pete rolls a six-sided die. What is the probability that the product of the two rolls is divisible by 3?

For the product to be a multiple of 3, one or both of the numbers appearing on the dice must be a multiple of 3. Exactly 2 of the numbers that could appear on each of the dice are multiples of 3. So the Inclusion-Exclusion Principle implies that there are

$$2 \cdot 6 + 8 \cdot 2 - 2 \cdot 2 = 24$$

ways that the product is a multiple of 3. Since there are $8 \cdot 6 = 48$ possible outcomes when the two dice are rolled, the probability of the product being a multiple of 3 is $24/48 = 1/2$.

3. What is the coefficient of $x^7$ in the expansion of

$$(1 + 2x - x^2)^4 ?$$

If we write

$$(1 + 2x - x^2)^4 = (1 + 2x - x^2) \cdot (1 + 2x - x^2) \cdot (1 + 2x - x^2) \cdot (1 + 2x - x^2),$$

we see that the highest power of $x$ is $x^8$, which occurs by choosing $-x^2$ from each of the four factors. To obtain the term $x^7$, we need to choose $-x^2$ from three of the factors and $2x$ for the fourth. There are four
distinct ways to make this choice, one for each factor, so the \( x^7 \) term in the expansion is

\[
4 \cdot (2x) \cdot (-x^2)^3 = -8x^7.
\]

OR

Although this is not a direct application of the Binomial Theorem, it produces the result

\[
(1 + 2x - x^2)^4 = ((1 + 2x) + (-x^2))^4
\]

\[
= (1 + 2x)^4 + 4(1 + 2x)^3(-x^2) + 6(1 + 2x)^2(-x^2)^2
\]

\[
+ 4(1 + 2x)(-x^2)^3 + (-x^2)^4.
\]

Only \( 4(1 + 2x)(-x^2)^3 = 4x^6 - 8x^7 \) has a term involving \( x^7 \), so the answer is \(-8\).

OR

The Zero-Coefficient Relationship for Polynomials implies that the coefficient of \( x^7 \) in the polynomial \( P(x) \) of degree 8 is the negative of the sum of the zeros of \( P(x) = (1 + 2x - x^2)^4 = (x^2 - 2x - 1)^4 \). Applying this same result to the polynomial \( x^2 - 2x - 1 \) implies that the sum of the zeros of this polynomial is the negative of the linear term, so the sum of the zeros of \( x^2 - 2x - 1 \) is 2. Hence the sum of the zeros of \( P(x) \) is

\[
4(2) = 8 \quad \text{and the coefficient of } x^7 \text{ is } -8.
\]

4. The number \( 25^{64} \cdot 64^{25} \) is the square of an integer \( N \). What is the sum of the digits of \( N \)?

This is a problem that can be resolved by first rearranging the calculations so that there are powers of 10. Here we have

\[
N = \left(25^{64} \cdot 64^{25}\right)^{1/2} = \left(\left(5^2\right)^{64}\right)^{1/2} \cdot \left(\left(2^6\right)^{25}\right)^{1/2}
\]

\[
= 5^{(2 \cdot 64 \cdot (1/2))} \cdot 2^{(6 \cdot 25 \cdot (1/2))}
\]

\[
= 5^{64} \cdot 2^{75} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048.
\]
Multiplying by $10^{64}$ does not effect sum of the digits of $N$, so the sum is $2 + 0 + 4 + 8 = 14$.

5. A pyramid has a square base and each edge of the pyramid has length 1. What is the volume of the pyramid?

Construct the right $\triangle ABC$ with one vertex, $A$, at one of the base vertices, a second, $B$, at the center of the base, and the third, $C$, at the top of the pyramid. Then $AB = \sqrt{2}/2$, $AC = 1$ and the height of the pyramid is

$$BC = \sqrt{1^2 - (\sqrt{2}/2)^2} = \sqrt{2}/2.$$ 

So the volume of the pyramid is

$$V = \frac{1}{3}(1)^2 \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}.$$ 

6. Suppose that for some integer $n$ the numbers $2n + 1$ and $3n + 1$ are perfect squares. Show that $n$ cannot be a prime number.

Suppose that $2n + 1 = a^2$ and $3n + 1 = b^2$. Then

$$n = 3n + 1 - (2n + 1) = b^2 - a^2 = (b - a)(b + a).$$

So $n$ cannot be prime unless $b - a = 1$. To see that this cannot be the case, suppose that $b = a + 1$. Then $n = b + a = 2a + 1$ and $2n + 1 = 4a + 3$. But $2n + 1$ is a perfect square and the square of an odd number must leave a remainder of 1 when divided by 4.

**OR**

If $b = a + 1$, then

$$3n + 1 = b^2 = (a + 1)^2 = a^2 + 2a + 1, \quad \text{and} \quad 2n + 1 = a^2,$$

so

$$\frac{1}{2}(a^2 - 1) = n = \frac{1}{3}(a^2 + 2a).$$

This equation in $a$ reduces to $a^2 - 4a - 3 = 0$, which has no integer solutions.
7. When a polynomial \( P(x) \) is divided by \( x - 19 \) the remainder is 99, and when \( P(x) \) is divided by \( x - 99 \) the remainder is 19. What is the remainder when \( P(x) \) is divided by \( (x - 19)(x - 99) \)?

Since \( (x - 19)(x - 99) \) is a quadratic polynomial, the remainder when this is divided into \( P(x) \) will be linear, that is,

\[
P(x) = (x - 19)(x - 99)Q(x) + ax + b,
\]

for some constants \( a \) and \( b \).

The Linear Factor Theorem implies that

\[
99 = P(19) = 19a + b \quad \text{and} \quad 19 = P(99) = 99a + b.
\]

Subtracting these equations and substituting gives

\[
80a = -80 \quad \text{so} \quad a = -1, \quad \text{and} \quad b = 99 - (-1)19 = 118.
\]

The remainder is therefore \(-x + 118\).

8. Let \( a \) and \( b \) be positive real numbers such that the quadratic equations

\[
x^2 + ax + 2b = 0 \quad \text{and} \quad x^2 + 2bx + a = 0
\]

both have real roots. What is the smallest possible sum for \( a + b \)?

These polynomials have real roots if and only if their discriminants are nonnegative, that is, if and only if

\[
a^2 - 4(2b) \geq 0 \quad \text{and} \quad (2b)^2 - 4a \geq 0.
\]

Hence

\[
b^2 \geq a \quad \text{and} \quad a^2 \geq 8b.
\]

So

\[
b^4 \geq a^2 \geq 8b \quad \text{and} \quad b^3 \geq 8, \quad \text{so} \quad b \geq 2.
\]

If \( b = 2 \), then \( 4 \geq a \). The pair \( a = 4 \) and \( b = 2 \) satisfies both \( b^2 \geq a \) and \( a^2 \geq 8b \), so the smallest possible sum is \( a + b = 4 + 2 = 6 \).

9. Let \( r_1, r_2, \) and \( r_3 \) be three real roots of the equation \( x^3 - 12x + 1 = 0 \). What is the value of

\[
\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1}?
\]
Obtaining a common denominator gives
\[
\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} = \frac{(r_2 + 1)(r_3 + 1) + (r_1 + 1)(r_3 + 1) + (r_1 + 1)(r_2 + 1)}{(r_1 + 1)(r_2 + 1)(r_3 + 1)}
\]
\[
= \frac{(r_1 r_2 + r_1 r_3 + r_2 r_3) + 2(r_1 + r_2 + r_3) + 3}{r_1 r_2 r_3 + 2(r_1 r_2 + r_1 r_3 + r_2 r_3) + 2(r_1 + r_2 + r_3) + 1}.
\]

The product of the roots is the negative of the constant term of \(x^3 - 12x + 1 = 0\), which is \(-1\). The sum of the roots is the negative of the coefficient of the \(x^2\) term, which is \(0\). And the sum of the product roots taken two at a time is the coefficient of the \(x\) term, which is \(-12\). Thus we have
\[
\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} = \frac{(r_1 r_2 + r_1 r_3 + r_2 r_3) + 2(r_1 + r_2 + r_3) + 3}{(-12) + 2(-1) + 1} = \frac{-11}{-24} = \frac{11}{24}.
\]

10. A circle that passes through two adjacent vertices of a square of side length 1 is also tangent to the opposite side of the square. What is the radius of the circle?

Suppose that the square is placed in the \(xy\)-plane with its vertices at \((0,0)\), \((1,0)\), \((0,1)\), and \((1,1)\), and that the circle passes through the vertices \((0,0)\) and \((1,0)\). Then the circle also passes through \((1/2,1)\), so the center must lie on the line \(x = 1/2\), at, say, \((1/2,y)\). Since \((0,0)\) and \((1/2,1)\) are equidistant from \((1/2,y)\) we have
\[
\left(\frac{1}{2}\right)^2 + y^2 = (1 - y)^2 = 1 - 2y + y^2.
\]
Hence
\[
\frac{1}{4} = 1 - 2y \quad \text{and} \quad y = \frac{3}{8}.
\]
This implies that the radius is
\[
\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{5}{8}.
\]