Challenge of Champions Test 2005

1. What is the value of \( k \) if
\[
2^{2007} - 2^{2006} - 2^{2005} + 2^{2004} = k \cdot 2^{2004}\ ?
\]

2. Joe rolls an eight-sided die and Pete rolls a six-sided die. What is the probability that the product of the two rolls is divisible by 3?

3. What is the coefficient of \( x^7 \) in the expansion of
\[
(1 + 2x - x^2)^4 \ ?
\]

4. The number \( 25^{64} \cdot 64^{25} \) is the square of an integer \( N \). What is the sum of the digits of \( N \)?

5. A pyramid has a square base and each edge of the pyramid has length 1. What is the volume of the pyramid?

6. Suppose that for some integer \( n \) the numbers \( 2^n + 1 \) and \( 3^n + 1 \) are perfect squares. Show that \( n \) cannot be a prime number.

7. When a polynomial \( P(x) \) is divided by \( x - 19 \) the remainder is 99, and when \( P(x) \) is divided by \( x - 99 \) the remainder is 19. What is the remainder when \( P(x) \) is divided by \( (x - 19) (x - 99) \)?

8. Let \( a \) and \( b \) be positive real numbers such that the quadratic equations
\[
x^2 + ax + 2b = 0 \quad \text{and} \quad x^2 + 2bx + a = 0
\]
both have real roots. What is the smallest possible sum for \( a + b \)?

9. Let \( r_1, r_2, \) and \( r_3 \) be three real roots of the equation \( x^3 - 12x + 1 = 0 \). What is the value of
\[
\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} \ ?
\]

10. A circle that passes through two adjacent vertices of a square of side length 1 is also tangent to the opposite side of the square. What is the radius of the circle?