Challenge of Champions Test 2004

1. How many distinct rearrangements of the letters in ABACUS has the last two letters as vowels?

2. A cubic polynomial $p(x)$ with leading coefficient 1, has value 8 at $-1$, $2$, and $3$. Find $p(5)$.

3. Let $L$ be the least common multiple of $a = 6!10!$ and $b = 7!9!$. Find $\frac{L^2}{ab}$.

4. Which real number $a$ gives the minimal value for the sum of the squares of the zeros of

   \[ x^2 - (a - 2)x - a - 1 = 0? \]

5. Show that if two adjacent angles of a quadrilateral are the right angles, then the bisectors of the other angles are perpendicular to each other.

6. A real function $f(x)$ that is not always 0 satisfies

   \[ xf(y) + yf(x) = (x + y)f(x)f(y) \]

   for all real $x$ and $y$. Find $f(2005)$.

7. Let $a$ and $b$ be positive real numbers. Show that

   \[ \frac{a^2 + b^2}{a + b} \geq \sqrt{ab}. \]

8. A man’s age at death in 1980 was $\frac{1}{29}$ of the year of his birth. How old did he turn in 1940?

9. Consider the sequence of integers:

   \[ a_0 = a_1 = 1, \quad a_{n+1} = a_n \cdot a_{n-1} + 1 \]

   for $n \geq 1$. Show that 4 does not divide $a_{2005}$.

10. Let $n$ be a positive integer. Show that all rational numbers of the form

    \[ \frac{21n + 4}{14n + 3} \]

    are in their lowest terms.