Challenge of Champions Test 2005:Solutions

1. What is the value of k if

   $$2^{2007} - 2^{2006} - 2^{2005} + 2^{2004} = k \cdot 2^{2004} \text{ ?}$$

   Since the exponent of the side with the unknown term k is 2004, we will express all the terms using this exponent. This gives

   $$k \cdot 2^{2004} = 2^{2004} \cdot 2^3 - 2^{2004} \cdot 2^2 - 2^{2004} \cdot 2 + 2^{2004} = (8-4-2+1)2^{2004} = 3 \cdot 2^{2004},$$

   so $k = 3$.

2. Joe rolls an eight-sided die and Pete rolls a six-sided die. What is the probability that the product of the two rolls is divisible by 3 ?

   For the product to be a multiple of 3, one or both of the numbers appearing on the dice must be a multiple of 3. Exactly 2 of the numbers that could appear on each of the dies are multiples of 3. So the Inclusion-Exclusion Principle implies that there are

   $$2 \cdot 6 + 8 \cdot 2 - 2 \cdot 2 = 24$$

   ways that the product is a multiple of 3. Since there are $8 \cdot 6 = 48$ possible outcomes when the two dice are rolled, the probability of the product being a multiple of 3 is $24/48 = 1/2$.

3. What is the coefficient of x^7 in the expansion of

   $$(1 + 2x - x^2)^4 \text{ ?}$$

   If we write

   $$(1+2x-x^2)^4 = (1+2x-x^2) \cdot (1+2x-x^2) \cdot (1+2x-x^2) \cdot (1+2x-x^2),$$

   we see that the highest power of x is x^8, which occurs by choosing $-x^2$ from each of the four factors. To obtain the term x^7, we need to choose $-x^2$ from three of the factors and $2x$ for the fourth. There are four distinct ways to make this choice, one for each factor, so the x^7 term in the expansion is

$$4 \cdot (2x) \cdot (-x^2)^3 = -8x^7.$$

OR

Although this is not a direct application of the Binomial Theorem, it produces the result

$$\begin{aligned}(1+2x-x^2)^4 &= \big((1+2x)+(-x^2)\big)^4 \\ &= (1+2x)^4 + 4(1+2x)^3(-x^2) + 6(1+2x)^2(-x^2)^2 \\ &\quad + 4(1+2x)(-x^2)^3 + (-x^2)^4.\end{aligned}$$

Only $4(1+2x)(-x^2)^3 = 4x^6 - 8x^7$ has a term involving x^7, so the answer is -8.

OR

The Zero-Coefficient Relationship for Polynomials implies that the coefficient of x^7 in the polynomial $P(x)$ of degree 8 is the negative of the sum of the zeros of $P(x) = (1+2x-x^2)^4 = (x^2-2x-1)^4$. Applying this same result to the polynomial x^2-2x-1 implies that the sum of the zeros of this polynomial is the negative of the linear term, so the sum of the zeros of x^2-2x-1 is 2. Hence the sum of the zeros of $P(x)$ is

$$4(2) = 8 \quad \text{and the coefficient of } x^7 \text{ is } -8.$$

4. The number $25^{64} \cdot 64^{25}$ is the square of an integer N. What is the sum of the digits of N ?

   This is a problem that can be resolved by first rearranging the calculations so that there are powers of 10. Here we have

$$\begin{aligned}N = \left(25^{64} \cdot 64^{25}\right)^{1/2} &= \left(\left(5^2\right)^{64}\right)^{1/2} \cdot \left(\left(2^6\right)^{25}\right)^{1/2} \\ &= 5^{(2 \cdot 64 \cdot (1/2))} \cdot 2^{(6 \cdot 25 \cdot (1/2))} \\ &= 5^{64} \cdot 2^{75} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048.\end{aligned}$$

Multiplying by 10^{64} does not effect sum of the digits of N, so the sum is $2+0+4+8=14$.

5. A pyramid has a square base and each edge of the pyramid has length 1. What is the volume of the pyramid?

   Construct the right $\triangle ABC$ with one vertex, A, at one of the base vertices, a second, B, at the center of the base, and the third, C, at the top of the pyramid. Then $AB=\sqrt{2}/2$, $AC=1$ and the height of the pyramid is
$$BC=\sqrt{1^2-(\sqrt{2}/2)^2}=\sqrt{2}/2.$$
   So the volume of the pyramid is
$$V=\frac{1}{3}(1)^2\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{6}.$$

6. Suppose that for some integer n the numbers $2n+1$ and $3n+1$ are perfect squares. Show that n cannot be a prime number.

   Suppose that $2n+1=a^2$ and $3n+1=b^2$. Then
$$n=3n+1-(2n+1)=b^2-a^2=(b-a)(b+a).$$
   So n cannot be prime unless $b-a=1$. To see that this cannot be the case, suppose that $b=a+1$. Then $n=b+a=2a+1$ and $2n+1=4a+3$. But $2n+1$ is a perfect square and the square of an odd number must leave a remainder of 1 when divided by 4.

OR

If $b=a+1$, then
$$3n+1=b^2=(a+1)^2=a^2+2a+1, \quad \text{and} \quad 2n+1=a^2,$$
so
$$\frac{1}{2}(a^2-1)=n=\frac{1}{3}(a^2+2a).$$
This equation in a reduces to $a^2-4a-3=0$, which has no integer solutions.

7. When a polynomial $P(x)$ is divided by $x - 19$ the remainder is 99, and when $P(x)$ is divided by $x - 99$ the remainder is 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

   Since $(x - 19)(x - 99)$ is a quadratic polynomial, the remainder when this is divided into $P(x)$ will be linear, that is,

   $$P(x) = (x - 19)(x - 99)Q(x) + ax + b, \quad \text{for some constants } a \text{ and } b.$$

   The Linear Factor Theorem implies that

   $$99 = P(19) = 19a + b \quad \text{and} \quad 19 = P(99) = 99a + b.$$

   Subtracting these equations and substituting gives

   $$80a = -80 \quad \text{so } a = -1, \quad \text{and} \quad b = 99 - (-1)19 = 118.$$

   The remainder is therefore $-x + 118$.

8. Let a and b be positive real numbers such that the quadratic equations

   $$x^2 + ax + 2b = 0 \quad \text{and} \quad x^2 + 2bx + a = 0$$

   both have real roots. What is the smallest possible sum for $a + b$?

   These polynomials have real roots if and only if their discriminants are nonnegative, that is, if and only if

   $$a^2 - 4(2b) \geq 0 \quad \text{and} \quad (2b)^2 - 4a \geq 0.$$

   Hence

   $$b^2 \geq a \quad \text{and} \quad a^2 \geq 8b.$$

   So

   $$b^4 \geq a^2 \geq 8b \quad \text{and} \quad b^3 \geq 8, \quad \text{so} \quad b \geq 2.$$

   If $b = 2$, then $4 \geq a$. The pair $a = 4$ and $b = 2$ satisfies both $b^2 \geq a$ and $a^2 \geq 8b$, so the smallest possible sum is $a + b = 4 + 2 = 6$.

9. Let r_1, r_2, and r_3 be three real roots of the equation $x^3 - 12x + 1 = 0$. What is the value of

   $$\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} + \frac{1}{r_3 + 1} \ ?$$

Obtaining a common denominator gives

$$\begin{aligned}\frac{1}{r_1+1}+\frac{1}{r_2+1}+\frac{1}{r_3+1} &= \frac{(r_2+1)(r_3+1)+(r_1+1)(r_3+1)+(r_1+1)(r_2+1)}{(r_1+1)(r_2+1)(r_3+1)}\\ &= \frac{(r_1r_2+r_1r_3+r_2r_3)+2(r_1+r_2+r_3)+3}{r_1r_2r_3+2(r_1r_2+r_1r_3+r_2r_3)+2(r_1+r_2+r_3)+1}.\end{aligned}$$

The product of the roots is the negative of the constant term of $x^3-12x+1=0$, which is -1. The sum of the roots in the negative of the coefficient of the x^2 term, which is 0. And the sum of the product roots taken two at a time is the coefficient of the x term, which is -12. Thus we have

$$\begin{aligned}\frac{1}{r_1+1}+\frac{1}{r_2+1}+\frac{1}{r_3+1} &= \frac{(r_1r_2+r_1r_3+r_2r_3)+2(r_1+r_2+r_3)+3}{r_1r_2r_3+2(r_1r_2+r_1r_3+r_2r_3)+2(r_1+r_2+r_3)+1}\\ &= \frac{(-12)+2(-1)+3}{(-1)+2(-12)+1}=\frac{-11}{-24}=\frac{11}{24}.\end{aligned}$$

10. A circle that passes through two adjacent vertices of a square of side length 1 is also tangent to the opposite side of the square. What is the radius of the circle?

    Suppose that the square is placed in the xy-plane with its vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and that the circle passes through the vertices $(0,0)$ and $(1,0)$. Then the circle also passes through $(1/2,1)$, so the center must lie on the line $x=1/2$, at, say, $(1/2,y)$. Since $(0,0)$ and $(1/2,1)$ are equidistant from $(1/2,y)$ we have

$$\left(\frac{1}{2}\right)^2+y^2=(1-y)^2=1-2y+y^2.$$

Hence

$$\frac{1}{4}=1-2y \quad\text{and}\quad y=\frac{3}{8}.$$

This implies that the radius is

$$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{3}{8}\right)^2}=\frac{5}{8}.$$